infinitesimal quantities. It cannot however be so used unless it can be shown that no error will result from such use in the particular case. In this case when $a = \infty$, u = actual zero or an infinitesimal, that is $u = 0 \div a$ or some infinitesimal between $1 \div a$ and $-1 \div a$, these infinitesimals depending for their values upon a and x. And the rate at which these infinitesimals change their values is $du \div dx = \cos ax$, for all values of a and x.

When a is infinite, that is greater than any assignable number, then a is indeterminately great, and by consequence indeterminate; $\cos ax$ is the cosine of an arc in a given circle, the arc being as indeterminate as a, and therefore $\cos ax$ is indetermiate both in "form" and value, and from the given conditions can no more be affirmed to be zero than any other value between $+1$ and -1. There being no preference or reason for the termination of the arc ax in one part of the circumference rather than another. Mathematics having to deal with truth, like the "Scripture is of no private interpretation".

DIFFERENTIATION OF THE LOGARITHM OF A VARIABLE.

BY PROF. LABAN E. WARREN, COLBY UNIV., WATERVILLE, ME.

To differentiate the logarithm of a variable, let $y = e^x$;

$$\therefore x = \log_e y; \ \therefore dx = d(\log_e y).$$

$$y + dy = e^{x+dx}, \ dy = e^{x+dx} - e^x \text{ or } dy = e^x(e^{dx} - 1),$$

$$e^{dx} = 1 + dx + \frac{dx^2}{2!} + \frac{dx^3}{3!} + \&c., \text{ or } e^{dx} = 1 + dx;$$

$$dy = e^x(1 + dx - 1), \text{ or } dy = e^x dx = y dx, \text{ or } dx = dy \div y,$$

but $dx = d(\log_e y)$;

$$\therefore d(\log_e y) = dy \div y, \text{ differential of Napierian logarithm.}$$

$$\log_{10} y = m(\log_e y); \ \therefore d(\log_{10} y) = m\, d(\log_e y);$$

$$\therefore d(\log_{10} y) = m\frac{dy}{y}, \text{ differential of common log.}$$

SOLUTIONS OF PROBLEMS NUMBER TWO.

Solutions of problems in No. 2 have been received as follows:

From Prof. L. G. Barbour, 434; Prof. W. P. Casey, 430, 431, 433; G. E. Curtis, 429, 434; Geo. Eastwood, 431; Wm. Hoover, 429, 430; Prof. P. H. Philbrick, 429, 430, 431, 433, 434; Prof. E. B. Seitz, 430, 431, 433, 435; Prof. J. Scheffer, 428, 430, 431, 433.

428. *By Geo. Lilley, A. M.* — "Two circles, of given radii, R and R_1, touch a straight line on the same side; a third circle of radius R_2 touches each of them; find the position of the circles R and R_1 and the radius of a fourth circle such that it shall touch the same straight line and each of the three given circles."

SOLUTION BY PROF. J. SCHEFFER.

Denote the radius of the required circle by x, the distance of its point of contact with the straight line from the points of contact of the circles R and R_1 with the straight line respectively by y and z. From the centre of the circle R_2 let fall the perpendicular u on the straight line, and denote the distance of its foot from the point of contact of the circle x with the straight line by t, then we have at once for the determination of the five quantities x, y, z, t, u the five equations:

$$(R+R_2)^2 = (y+t)^2+(u-R)^2, \qquad (1)$$

$$(R_1+R_2)^2 = (y-t)^2+(u-R_1)^2, \qquad (2)$$

$$(R+x)^2 = y^2+(R-x)^2, \text{ whence } y^2 = 4Rx, \qquad (3)$$

$$(R_1+x)^2 = z^2+(R_1-x)^2, \quad \text{“} \quad z^2 = 4R_1x, \qquad (4)$$

$$(R_2+x)^2 = t^2+(u-x)^2, \quad \text{“} \quad t^2+u^2 = R_2^2+2R_2x+2ux. \qquad (5)$$

Expanding (1) and (2) and substituting the values of y, z and t^2+u^2 from (3), (4), (5) we obtain the two equations:

$$2\sqrt{(Rx)}\times t+(x-R)u = RR_2-2Rx-R_2x, \qquad (6)$$

$$2\sqrt{(R_1x)}\times t+(x-R_1)u = R_1R_2-2R_1x-R_2x, \qquad (7)$$

whence, after some easy reductions:

$$t = \frac{(\sqrt{R}-\sqrt{R_1})x^2}{[\sqrt{(RR_1)}-x]\sqrt{x}}, \quad u = \frac{R_2x+2x\sqrt{(RR_1)}-R_2\sqrt{(RR_1)}}{\sqrt{(RR_1)}-x}. \qquad (8)$$

Substituting these values of t and u in (5), we obtain the quadratic eq'n

$$(\sqrt{R}+\sqrt{R_1})^2.x^2+4R_2\sqrt{(RR_1)}.x = 4RR_1R_2,$$

whence

$$x = 2\sqrt{(RR_1R_2)}\times\frac{-\sqrt{R_2}+\sqrt{[R+R_1+R_2+2\sqrt{(RR_1)}]}}{(\sqrt{R}+\sqrt{R_1})^2}.$$

The values of y, z, t, u can now be derived respectively from (3), (4), (8). If $R_2 = R_1 = R$, we get $x = \frac{1}{2}(\sqrt{5}-1).R$.

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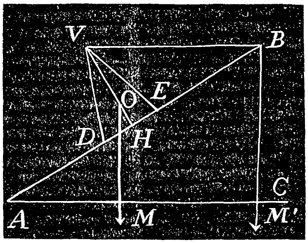
429. *By Prof. M. L. Comstock.*—"A cone of given weight W, is placed with its base on an inclined plane, and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pulley in the inclined plane at the same height as the vertex. Determine the conditions of equilibrium."

SOLUTION BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY.



Let $i = \angle BAC$, $r = EH =$ radius of base of cone, $nr =$ altitude of cone, $\mu =$ coefficient of friction, $N =$ normal pressure of the cone on the plane, and $F =$ friction between the cone and the plane.

Resolving forces, normal to the plane,

$N = W' \sin i + W \cos i$; $\therefore$

$F = \mu N = \mu(W' \sin i + W \cos i)$; (1)

parallel to the plane, $F + W' \cos i - W \sin i) = 0$. (2)

From (1) and (2) we get

$$\tan i = \frac{\mu W + W'}{W - \mu W'}. \tag{3}$$

Taking moments about E, observing that $OH = \frac{1}{4}\, VH$, we get

$W'(nr \cos i - r \sin i) =$ or $< W(r \cos i + \frac{1}{4} nr \sin i)$, or

$W'(n - \tan i) =$ or $< W(1 + \frac{1}{4} n \tan i)$, (4)

$$n \begin{matrix} = \\ < \end{matrix} \frac{W + W' \tan i}{W' - \frac{1}{4} W \tan i}. \tag{5}$$

If $W = 0$ (5) gives $n =$ or $< \tan i$, or $nr =$ or $< r \tan i = HD$, and the string passes through E or some point between E and H.

If $W' = 0$, $n =$ or $< -(4 \div \tan i)$ or $\frac{1}{4} nr \tan i =$ or $< -r = HD$ and the perpendicular through the center of gravity of the cone must pass thro' D. If $n = \infty$, that is, if the base of the cone is inappreciable in comparison with the height, either (4) or (5) gives $W' = \frac{1}{4} W \tan i$, or $W' \times nr \cos i = W \times \frac{1}{4} nr \sin i$, that is the moments of W' and W about H are equal.

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430 *By Prof. Milwee, Add-Ran Col. Texas.*—"Given two fixed points A and B, one on each of the axes of coordinates, at the respective distances a and b from the origin; if A' and B' be taken on the axes so that $OA' + OB' = OA + OB$, find the locus of the intersection of AB' and $A'B$."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $AO = a$, $BO = b$, $A'O = a + p$, then will $B'O = b - p$, from the conditions of the problem.

The equations of AB' and $A'B$ are respectively,

$$\frac{x}{a} + \frac{y}{b-p} = 1, \qquad \frac{x}{a+p} + \frac{y}{b} = 1,$$

or $\quad bx + ay - ab + p(a - x) = 0,\ bx + ay - ab + p(y - b) = 0.$

Subtracting, $p(a - x) = p(y - b)$, or $x + y = a + b$, the eq. of the locus.

431. *By Prof J. W. Nicholson.*—"Required the area of a triangle whose sides are equal to the three roots respectively of the following equation:

$$x^3 + mx^2 + nx + r = 0." \qquad (1)$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let a, b, c be the sides of the triangle. The equation whose roots are a, b, c is $(x-a)(x-b)(x-c) = 0$, or

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0. \qquad (2)$$

To make (1) and (2) identical we must have

$$a + b + c = -m, \qquad (3)$$

$$ab+ac+bc = n, \qquad (4)$$

$$abc = -r. \qquad (5)$$

Subtracting (4)×2 from the square of (3), we have

$$a^2 + b^2 + c^2 = m^2 - 2n. \qquad (6)$$

Subtracting (3)×(5)×2 from the square of (4), we have

$$a^2b^2+a^2c^2+b^2c^2 = n^2-2mr. \qquad (7)$$

Subtracting the square of (6) from (7)×4, we have

$$2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 = 4m^2n-8mr-m^4. \qquad (8)$$

But the first member of (8) is equal to 16 times the square of the area of the triangle. Therefore we have

$$\text{area of triangle} = \tfrac{1}{4}\sqrt{(4m^2n - 8mr - m^4)}.$$

[This problem was solved in the same way, and the same result obtained, by Prof. Scheffer.]

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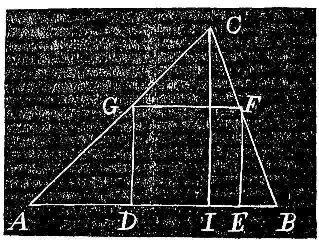
432. No solution received.

433. *By Prof. W. P. Casey.*—"Given the base of a triangle, to find the locus of the vertex, when the centre of the inscribed square moves on a given conic section."

SOLUTION BY PROF. J. SCHEFFER, HARRISBURG, PA.



Denote the given base AB by a. Let A be the origin of coordinates and AB the axis of x, and denote AI and CI respectively by x and y, and the side of the inscribed square by z. We easily obtain

$$z = \frac{ay}{a+y}, \text{ and } AD = \frac{ax}{a+y};$$

therefore the coordinates of the centre of the square are

$$AD+\tfrac{1}{2}z = \frac{2ax+ay}{2(a+y)}\ \ldots (1), \text{ and } \tfrac{1}{2}z = \frac{ay}{2(a+y)}. \qquad (2)$$

The values (1) and (2) substituted in the equation of the given curve will produce the equation of the locus required. Substituting them in the general equation of a conic $Ay^2+Bxy+Cx^2+Dy+Ex+F=0$, we obtain the equation of the locus,

$$(Aa^2+Ba^2+Ca^2+2Da+2Ea+4F)y^2+(2Ba^2+4Ca^2+4Ea)xy+4Ca^2x^2$$
$$+(2Da^2+2Ea^2+8Fa)y+4Ea^2x+4Fa^2 = 0,$$

which is the equation of a conic section, and is an ellipse, hyperbola or parabola according as $(Ba+2E)^2$ is $<$, $>$, or $= 4C(Aa^2+2Da+4F)$.

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434. *By Prof. De Volson Wood.*—"Find a number, the mantissa of the logarithm of which equals the number."

SOLUTION BY PROF. PHILBRICK.

If p is small and m the modulus of a system of logarithms, we have, approximately, $\log(1-p) = -mp = -1+(1-mp)$, $1-mp$ being the mantissa.

Now $(1-mp)-(1-p) = p(1-m) = e$ say, which approaches 0 as p approaches 0, or $1-p$ approaches 1; hence unity is the number sought.

In the common system $m = .4343$ and $e = .5657p$; in the Napierian system $m=1$ and $e=0$.

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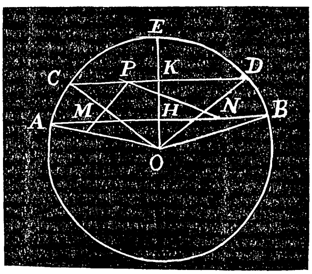
435. *By Prof. E. B. Seitz.*—"Find the average area of a triangle drawn on the surface of a given circle, having its base parallel to a given line, and its vertex taken at random."

SOLUTION BY THE PROPOSER.



Let O be the center of the given circle, and MNP a triangle drawn on the surface of the circle, having the side MN parallel to a given line.

Produce MN, forming the chord AB; draw the chord CD through P parallel to AB, and draw the radius OE perpendicular to AB.

Let $OA = r$, $BM = x$, $MN = y$, $\angle AOE = \theta$, and $\angle COE = \varphi$. Then $AB = 2r\sin\theta$, $CD = 2r\sin\varphi$, $OH = r\cos\theta$, $OK = r\cos\varphi$,

area $MNP = \frac{1}{2}MN \times HK = \frac{1}{2}ry(\cos\varphi - \cos\theta) = u$, when $\varphi < \theta$,

" " " $= \frac{1}{2}ry(\cos\theta - \cos\varphi) = u_1$, when $\varphi > \theta$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and θ and π; of x, 0 and $2r\sin\theta = x'$; and of y, 0 and x. Therefore the required average is

$$\frac{\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta 2ur^2\sin^2\varphi d\varphi + \int_\theta^\pi 2u_1r^2\sin^2\varphi d\varphi\right\}\int_0^{x'}\int_0^x r\sin\theta\, d\theta dx dy}{\int_0^{\frac{1}{2}\pi}\int_0^\pi\int_0^{x'}\int_0^x r\sin\theta . 2r^2\sin^2\varphi\, d\varphi dx dy}$$

$$= \frac{3}{2\pi r}\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta(\cos\varphi - \cos\theta)\sin^2\varphi d\varphi\right.$$

$$\left. + \int_\theta^\pi(\cos\theta - \cos\varphi)\sin^2\varphi d\varphi\right\}\int_0^{x'}\int_0^x \sin\theta d\theta dx dy$$

$$= \frac{2r^2}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta(\cos\varphi - \cos\theta)\sin^2\varphi d\varphi + \int_\theta^\pi(\cos\theta - \cos\varphi)\sin^2\varphi\, d\varphi\right\} \times \sin^4\theta d\theta$$

$$= \frac{r^2}{3\pi}\int_0^{\frac{1}{2}\pi}\Big[3(\pi - 2\theta)\cos\theta + 4\sin\theta + 2\sin\theta\cos^2\theta\Big]\sin^4\theta d\theta = \frac{512r^2}{525\pi}.$$

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436. No solution received.

NOTE BY W. E. HEAL.—The values of A, B, C, &c., given by me at page 56 (ANALYST, No. 2) should be

$A = b^8[a^4 - (a^2+b^2)x^2]^2\tan^2\varphi - 4a^4b^{10}x^2 + 4a^2b^{10}x^4.$

$B = 4[a^2b^6(a^2+b^2)^2x^3\lambda - a^6b^6(a^2+b^2)x\lambda]\tan^2\varphi - 8a^6b^8x\lambda + 16a^4b^8x^3\lambda.$

$C = [2a^8b^8 - 2a^4b^4(a^2+b^2)(a^4\lambda^2+b^4x^2) + 6a^4b^4(a^2+b^2)^2x^2\lambda^2]\tan^2\varphi$
$\qquad - 4(a^8b^6\lambda^2 + a^6b^8x^2) + 24a^6b^6x^2\lambda^2.$

$D = 4[a^6b^2(a^2+b^2)^2x\lambda^3 - a^6b^6(a^2+b^2)x\lambda]\tan^2\varphi - 8a^8b^6x\lambda + 16a^8b^4x\lambda^3.$

$E = a^8[b^4 - (a^2+b^2)\lambda^2]^2\tan^2\varphi - 4a^{10}b^4\lambda^2 + 4a^{10}b^2\lambda^4.$

Also in lines 2 and 9 of same page "a^2y" should be a^2y^2.

PROBLEMS.

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437. *By Prof. Casey.*—AE, AK are two indefinite given straight lines, C and H given points in them, and P a given point in their plane. Req'd to draw through P two straight lines, PB, PD, intersecting AE and AK in R and S, respectively, and containing a given angle RPS, so that $RC \times SH$ may be equal to a given magnitude.